*On Two Related Transformations of Space Curves.**

By William Caspar Graustein

The Transformation C.

1. Two curves, corresponding point for point so that the tangents in corresponding points are parallel, are said to be related by a transformation of Combescure.† The general curve in 1—1 correspondence with a given space curve, $x=x(s)$,‡ may be given by $y=x+f\alpha+\phi\beta+\psi\gamma$, where f, ϕ, ψ are functions of s. The conditions that the tangents in corresponding points be parallel are

$$\frac{d\phi}{ds}+\frac{f}{R}+\frac{\psi}{T}=0, \quad \frac{d\psi}{ds}-\frac{\phi}{T}=0.$$

If there is set $\psi=\dfrac{dC}{ds_\gamma}$, then $\phi=-\dfrac{d^2C}{ds_\gamma^2}$, $f=\dfrac{d}{ds_\alpha}\left(C+\dfrac{d^2C}{ds_\gamma^2}\right)$, and

$$y=x+\delta, \text{ where } \delta=\frac{d}{ds_\alpha}\left(C+\frac{d^2C}{ds_\gamma^2}\right)\alpha-\frac{d^2C}{ds_\gamma^2}\beta-\frac{dC}{ds_\gamma}\gamma. \tag{1}$$

This y-curve is the general curve in Combescurian correspondence with the x-curve. Since it is determined, when the function C (except for an additive constant) is given, and conversely, one may speak of the correspondence, or transformation, C. If $C=R$, the y-curve is the second polar curve of the x-curve (the first polar curve of a curve is the edge of regression of the polar developable, the second is the polar curve of the first, etc.). Now

$$\frac{dy}{ds}=\left(1+\frac{P(C)}{R}\right)\alpha, \text{ where } P(C)=\frac{d^2}{ds_\alpha^2}\left(C+\frac{d^2C}{ds_\gamma^2}\right)+\frac{d^2C}{ds_\gamma^2}\,.$$

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* Read before the American Mathematical Society, December 28, 1915.

† Aoust discusses this transformation in his "Analyse infinitesimale des courbes dans l'espace," 1876, Ch. XVI. It comes by its present title, the writer believes, from its similarity with the Combescurian transformation of triply orthogonal systems of surfaces. Salkowski gives a geometrical treatment of it in the *Mathematische Annalen*, Vol. LXVI (1909), and Sannia gives an analytical one, by means of the moving trihedral, in the *Rendiconti del Circolo Matematico di Palermo*, Vol. XX (1905), pp. 83-92.

‡ In this paper x_1, x_2, x_3 are the coordinates of a point x; $x_1=x_1(s)$, $x_2=x_2(s)$, $x_3=x_3(s)$ is the parametric representation, in terms of the arc s, of the curve $x=x(s)$; α_1, α_2, α_3, β_1, β_2, β_3, γ_1, γ_2, γ_3, are the direction cosines of the directions α, β, γ of the tangent, principal normal and binormal; R and T are the radii of curvature and torsion; s_α and s_γ are the arcs of the tangent and binormal indicatrices. Plane curves are excluded and the work is for the real domain.

Thus $\alpha_y = \pm\alpha$, and hence $\beta_y = \pm\beta$; if the choices $\alpha_y = \alpha$, $\beta_y = \beta$ are made, it follows that $\gamma_y = \gamma$. Further

$$\frac{ds_y}{ds} = \frac{R_y}{R} = \frac{T_y}{T} = 1 + \frac{P(C)}{R}. \tag{2}$$

Thus the rectifying lines in corresponding points are parallel also.

The surface, $z = x + r\delta$, generated by the line xy, is a developable. Its edge of regression, if its exists, is given by $\eta = x - R/P(C)\,\delta$.

2. The x- and y-curves are congruent* only if $P(C) = 0$. Now

$$P(C) = \left(\frac{ds_\gamma}{ds_\alpha}\right)^2 \Gamma\left(\frac{dC}{ds_\gamma}\right), \tag{3}$$

where $\Gamma(\gamma) = 0$ is the result of elimination of α and β from the Frenet formulae. The general solution of $\Gamma(\gamma) = 0$ is then $-r(c|\gamma)$,† where $(c|c) = 1$ and $r > 0$; hence the general solution of $P(C) = 0$ is

$$C = -r\int (c|\gamma)\,ds_\gamma, \quad \text{or} \quad \frac{dC}{ds_\gamma} = -r(c|\gamma). \tag{4}$$

Substitution of this value of C in (1) yields $y = x + rc$.

Thus the general transformation C, carrying a given curve into a congruent curve, is defined by (4) and is equivalent to a translation in the direction c through the distance r. On the other hand, if the transformation C is given, (4) becomes the defining relation for the curves congruent to their transforms under it. A parametric representation of these curves, as the curves defined by (4) and in Combescurian correspondence with an arbitrarily given curve, the ξ-curve, may be found by a method of Salkowski.‡ Since $\gamma = \gamma_\xi$, $ds_\gamma = (ds_\gamma)_\xi$, the first of the formulae (4) may be written

$$C = -r\int (c|\gamma_\xi)\,(ds_\gamma)_\xi. \tag{5}$$

If C involves s, R and T, (5) and the last of the equations

$$\frac{ds}{(ds)_\xi} = \frac{R}{R_\xi} = \frac{T}{T_\xi} \tag{6}$$

may be solved for R and T as functions of s and the parameter of the ξ-curve,

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* In the *Bulletin de la Société Mathématique*, Vol. VII (1878), pp. 143–154, Aoust set up a differential equation of the curves congruent to their second polar curves, which Hoppe recognized later, *Archiv der Mathematik und Physik*, Vol. LXVI (1881), pp. 380–396, as that which the direction cosines of the binormal of a space curve satisfy, and thus easily integrated. It is this idea of Hoppe that is being used here in the case of the general transformation C.

† N. B. $(c|\gamma) = c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3$, $(c|c) = c_1^2 + c_2^2 + c_3^2$.

‡ *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, Vol. IV (1905), pp. 64–69.

and thus either of the remaining equations of (6) may be solved for s as a function of the parameter of the ξ-curve. Hence

$$x=\int \frac{ds}{(ds)_\xi}\, d\xi, \quad \text{or} \quad x=\int \frac{R}{R_\xi}\, d\xi, \quad \text{or} \quad x=\int \frac{T}{T_\xi}\, d\xi$$

yields a parametric representation of the required curves. If, in particular, $C=R\,F\left(\frac{R}{T}\right)$, this method results in the representation

$$x=r\int \frac{\int (c\,|\,\gamma_\xi)\,(ds_\gamma)_\xi}{C_\xi}\, d\xi.$$

For $C=R$ this becomes the parametric representation of the curves congruent to their second polar curves.

The y-curve is similar to the x-curve,* if R_y and T_y are constant multiples of R and T; that is, by (2), if $P(C)=kR$, where k is a constant, not -1. By (3) this condition becomes

$$\Gamma\left(\frac{dC}{ds_\gamma}\right)=kR\left(\frac{ds_a}{ds_\gamma}\right)^2.$$

The general solution of this equation is

$$\frac{dC}{ds_\gamma}=r\,(c\,|\,\gamma)-k\,(x\,|\,\gamma). \tag{7}$$

For this value of C (1) may be written ($k\neq 0$)

$$y-\frac{rc}{k}=(1+k)\left(x-\frac{rc}{k}\right).$$

Thus the general transformation C, carrying a given curve into a similar curve, is defined by (7) and is equivalent to a stretching from the point rc/k with coefficient of enlargement $1+k$. Work similar to that leading to this result yields a new representation of the general y-curve: The general transformation C, carrying a given x-curve into a y-curve such that R_y and T_y are given multiples of R and T—$R_y/R=T_y/T=1+K$, where K is a given function, $\neq -1$—is defined by $dC/ds_\gamma=r\,(c\,|\,\gamma)-(\gamma\,|\int K\,dx)$, and the most general such y-curve is then $y=x+\int K\,dx-rc$.

The curves similar to their transforms under a given transformation C are defined by (7). The problem of setting up a parametric representation for these curves is, in general, incapable of explicit solution.†

* Salkowski discusses this problem for a curve and its second polar curve, *Archiv der Mathematik und Physik*, Vol. XIV (1908), pp. 231–239.

† For the transformation $C=R$, it depends on the solution of three simultaneous linear differential equations of the second order.

3. If the x- and y-curves are congruent, the distance $xy=d$ is constant. The converse, however, is not true;* differentation of $d^2=(\delta|\delta)$ with respect to s yields the product of $2P(C)$ and

$$\frac{d}{ds_\alpha}\left(C+\frac{d^2C}{ds_\gamma^2}\right). \tag{8}$$

Thus the transformation $C=c_0+c\cdot\sin(s_\gamma+k)$, $c\neq 0$, applied to any given x-curve, yields a non-congruent y-curve at a constant distance. The representation of the curves, which transform into non-congruent curves at constant distances under a given transformation C, may be found by Salkowski's method. Thus the curves, which are at constant distances from their second polar curves but not congruent to them, are those whose first polar curves are of constant curvature, while they themselves are not.

Whereas constant length of xy is not a sufficient condition for the congruence of the x- and y-curves, constant direction of xy is. This may be shown by differentiating the direction cosines δ/d of xy.

The direction xy is relatively fixed with respect to the trihedral in x only if the factors of combination of α, β, γ in δ/d are constant. This is true only if the x-curve is a helix and C is such that $dC/ds_\gamma=ke^{ms_\gamma}$. If $m=1$, the two curves are congruent helices on the same cylinder.

4. The transformations C, such that the point tracing one of the curves lies always in the corresponding rectifying plane of the other, are given by $dC/ds_\gamma=a$, a constant; then (1) becomes $y=x+a(\alpha R/T-\gamma)$. Thus the two curves are parallel geodesics on the developable z; conversely, two parallel geodesics on an arbitrary developable are related by such a transformation C.

The curves whose rectifying planes contain the corresponding points of their second polar curves are those for which the distance from the center of curvature to the center of the osculating sphere is constant. The only such curves, which, further, are congruent to their second polar curves are helices for which $R^2=bs$; these helices lie on cylinders with involutes of circles as directrices.

The transformations C, such that the normal plane of one curve contains always the corresponding point of the other, are given by (8), set equal to zero. The two curves are orthogonal trajectories of the rulings of the developable z; conversely, two orthogonal trajectories of the rulings of an arbitrary developable are related by such a transformation C. The curves whose normal

* Salkowski supposes the converse true in his geometric determination of the curves congruent to their second polar curves, *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, Vol. VI (1907), pp. 54–59.

planes contain the corresponding points of their second polar curves are those whose first polar curves are of constant curvature.

The only y-curve in correspondence C with an x-curve such that y always lies in the osculating plane of x is the x-curve itself.

The Transformation A.

5. The transformation of Combescure finds an analytic parallel in the point-to-point correspondence of two curves, in which the tangents in corresponding points have as their common perpendicular direction that of the principal normal of the given curve and make with one another an angle whose cosine equals the ratio of the corresponding elements of arc—so that the element of arc of the given curve is the projection of that of the transformed curve. The y-curve is the transform of a given x-curve by this transformation, if $(\alpha_y|\beta)=0$, $(\alpha_y|\alpha)ds_y=ds$; hence it has the general form

$$y=x+\delta, \text{ where } \delta=-\frac{dA}{ds_\alpha}\alpha-\frac{d^2A}{ds_\alpha^2}\beta+\frac{d}{ds_\gamma}\left(A+\frac{d^2A}{ds_\alpha^2}\right)\gamma. \tag{9}$$

When the function A (except for an additive constant) is given, this y-curve is determined, and, conversely. Thus one may speak of the correspondence, or transformation, A.

A comparison of (9) with (1) shows that analytically the transformation A becomes the transformation C if α and γ, ds_α and ds_γ, R and T, are interchanged. This relationship between the two transformations subsists to a great extent throughout. Thus, from (9),

$$\frac{dy}{ds}=\alpha+\frac{Q(A)}{T}\gamma, \text{ where } Q(A)=\frac{d^2}{ds_\gamma^2}\left(A+\frac{d^2A}{ds_\alpha^2}\right)+\frac{d^2A}{ds_\alpha^2}\,; \tag{10}$$

$Q(A)$ becomes $P(C)$ by replacing A by C and interchanging ds_α and ds_γ.

If θ, where $\tan\theta=Q(A)/T$, is introduced, then for the y-curve

$$\left.\begin{aligned}
\alpha_y&=\cos\theta\,\alpha+\sin\theta\,\gamma,\\
\sqrt{\theta'^2+M^2}\beta_y&=\quad M\beta+\theta'(\cos\theta\,\gamma-\sin\theta\,\alpha),\\
\sqrt{\theta'^2+M^2}\gamma_y&=-\theta'\beta+M(\cos\theta\,\gamma-\sin\theta\,\alpha),\\
\frac{1}{R_y}&=\cos\theta\sqrt{\theta'^2+M^2},\\
\frac{1}{T_y}&=\cos\theta\frac{\theta'M'-M\theta''-N\theta'^2-M^2N}{\theta'^2+M^2},
\end{aligned}\right\} \tag{11}$$

where
$$M=\frac{\cos\theta}{R}+\frac{\sin\theta}{T},\quad N=\frac{\sin\theta}{R}-\frac{\cos\theta}{T},$$

and the primes denote differentiation with respect to s.

The ruled surface, $z=x+r\delta$, generated by the line xy, is a developable only when $Q(A)=0$ or $A=as_a+b$. The common perpendicular direction to two neighboring rulings of the surface is parallel always to the normal plane of the x-curve. The x-curve is the line of striction on the surface only if $d^2A/ds_a^2+A=$const.

6. If the x- and y-curves are to be, further, in Combescurian correspondence, it is evident from (10), that $Q(A)=0$; but then the curves are obviously congruent. Now

$$Q(A)=\left(\frac{ds_a}{ds_\gamma}\right)^2\Delta\left(\frac{dA}{ds_a}\right), \tag{12}$$

where $\Delta(\alpha)=0$ is the differential equation of the direction cosines of the direction α and thus has, as its general solution, $-r(a\,|\,\alpha)$, where $(a\,|\,a)=1$, $r>0$. Hence the general solution of $Q(A)=0$ is

$$\frac{dA}{ds_a}=-r(a\,|\,\alpha). \tag{13}$$

Substitution of this value in (9) gives $y=x+ra$.

The general transformation A, carrying a given curve into a congruent curve, is defined by (13) and is equivalent to a translation in the direction a through the distance r. On the other hand, if the transformation A is given, (13) becomes the defining relation for the curves congruent to their transforms under it. A parametric representation of these curves may be obtained by Salkowski's method.

7. The ratio of the elements of arc of the x- and y-curves is constant, or, what is equivalent, the angle θ formed by the directions of their tangents is constant, only if $Q(A)/T(=\tan\theta)$ is constant. By (12) this condition becomes

$$\Delta\left(\frac{dA}{ds_a}\right)=\tan\theta\, T\left(\frac{ds_\gamma}{ds_a}\right)^2.$$

The general solution of this eqnation is

$$\frac{dA}{ds_a}=r(a\,|\,\alpha)-\tan\theta(\textstyle\int\gamma ds\,|\,\alpha). \tag{14}$$

For this value of dA/ds_a, (9) reduces to $y=x+\tan\theta\int\gamma ds-ra$.

The corresponding elements of arc of the x-curve and its transform by A are in constant ratio, or the directions of their corresponding tangents form a constant angle θ, only when A is given by (14). Work similar to that leading

to this result yields a new means of representing the general y-curve: The general transformation A, carrying a given x-curve into a y-curve whose element of arc is a given multiple of that of the x-curve—$ds_y=\sec\theta ds$, where θ is a given function—is defined by $dA/ds_a=r(a\,|\,\alpha)-(\int\tan\theta\,\gamma ds\,|\,\alpha)$, and the most general such y-curve is then $y=x+\int\tan\theta\,\gamma ds-ra$.

If θ is constant, the formulae (11) become

$$\alpha_y=\cos\theta\,\alpha+\sin\theta\,\gamma,\quad \beta_y=\beta,\quad \gamma_y=\cos\theta\,\gamma-\sin\theta\,\alpha,$$
$$\frac{\sec\theta}{R_y}=\frac{\cos\theta}{R}+\frac{\sin\theta}{T},\quad \frac{\sec\theta}{T_y}=\frac{\cos\theta}{T}-\frac{\sin\theta}{R},\quad ds_y=\sec\theta\,ds.$$

Thus, if the y-curve results from the x-curve by a transformation A, preserving the ratio of the elements of arc, then the principal normals of the two curves are parallel, and conversely; their rectifying lines are parallel and the elements of arc of their principal normal indicatrices are in constant ratio. Further, if the given curve is a curve of constant curvature or torsion, the transformed curve is a curve of Bertrand; conversely, a curve of Bertrand may be carried over by a suitable such transformation A into a curve of constant curvature or torsion. (Thus the determination of the curves of Bertrand is reduced to the determination of the curves of constant curvature or torsion.) Finally, under such a transformation A, a helix goes into a helix and a circular helix into a circular helix.

The surface z is, for constant θ, $z=x+\rho(\tan\theta\int\gamma ds-ra)$. The curve $\rho=1$ on it is the transform of the x-curve by (14). Evidently an arbitrary curve $\rho=$const. is also such a transform. Thus the curves $\rho=$const. have their principal normals along a ruling of the surface parallel. Conversely, if a ruled non-developable surface contains a family of curves having parallel principal normals along each ruling, these curves are the transforms of one of their number by transformations of the type (14).

8. The transformations A, such that the point tracing the y-curve lies always in the corresponding rectifying plane of the x-curve, are given by $dA/ds_a=a$, a constant; the y-curve is then $y=x-a(\alpha-T/R\,\gamma)$. Evidently the surface z is the rectifying developable of the x-curve and thus the x-curve is a geodesic on it. Conversely, a geodesic on a developable may be carried into any second curve on the developable, whose tangent is perpendicular to the corresponding principal normal of the geodesic, by such a transformation A. The y-curve is also a geodesic on the developable only when the x-curve is a helix; the two curves are then congruent.

The transformations A, such that the osculating plane of the y-curve contains always the corresponding point of the x-curve, are given by

$$\frac{dA}{ds_a} = c \cos (s_a + k). \tag{15}$$

The y-curve is then $y = x + \delta$, where $\delta = -c[\cos (s_a + k)\alpha - \sin(s_a + k)\beta]$. The x-curve is an asymptotic line and also the line of striction on the surface $z = x + r\delta$. The curves $r =$ const. on the surface are the curves at constant distances, along the rulings, from the x-curve and each of them is a transform of it by a suitable transformation (15). Conversely, if the line of striction of a ruled surface is asymptotic, the curves at constant distances from it are its transforms by transformations A of the form (15). If these curves are to be asymptotic also, the line of striction must be a curve of the category defined by $dT/ds_\gamma = c \cos (s_a + k)$.

The only y-curve in correspondence A with an x-curve so that y always lies in the normal plane of x is the x-curve itself.

9. Here again, as in the case of the transformation C, a constant distance $xy = d$ is not sufficient for the congruence of the x- and y-curves. In fact, the distance d is constant, and the x- and y-curves are still not congruent, in case of the special transformation (15).

On the other hand, and again as in the case of the transformation C, constant direction of xy is a sufficient condition for congruence of the x- and y-curves.

If the direction xy is relatively fixed with respect to the trihedral in x, the x-curve is a helix and the transformation is given by $dA/ds_a = ke^{ms_a}$, and conversely.

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